

# A CHARACTERISATION OF VIRTUALLY FREE GROUPS

ROBERT H. GILMAN, SUSAN HERMILLER, DEREK F. HOLT, AND SARAH REES

**ABSTRACT.** We prove that a finitely generated group  $G$  is virtually free if and only if there exists a generating set for  $G$  and  $k > 0$  such that all  $k$ -locally geodesic words with respect to that generating set are geodesic.

**Keywords:** Virtually free group; Dehn algorithm; word problem.

**Mathematics Subject Classification:** 20E06; Secondary 20F67.

## 1. INTRODUCTION

A group is called virtually free if it has a free subgroup of finite index.

In this article we characterise finitely generated virtually free groups by the property that a Dehn algorithm reduces any word to geodesic form. Equivalently, a group is virtually free precisely when the set of  $k$ -locally geodesic words and the set of geodesic words coincide for suitable  $k$  and appropriate generating set.

Let  $G$  be a group with finite generating set  $X$ . We shall assume throughout this article that all generating sets of groups are closed under the taking of inverses. For a word  $w = x_1 \cdots x_n$  over  $X$ , we define  $l(w)$  to be the length  $n$  of  $w$  as a string, and  $l_G(w)$  to be the length of the shortest word representing the same element as  $w$  in  $G$ . Then  $w$  is called a *geodesic* if  $l(w) = l_G(w)$ , and a  *$k$ -local geodesic* if every subword of  $w$  of length at most  $k$  is geodesic.

Let  $\mathcal{R}$  be a finite set of length-reducing rewrite rules for  $G$ ; that is, a set of substitutions

$$u_1 \rightarrow v_1, u_2 \rightarrow v_2, \dots, u_r \rightarrow v_r,$$

where  $u_i =_G v_i$  and  $l(v_i) < l(u_i)$  for  $1 \leq i \leq r$ . Then  $\mathcal{R}$  is called a *Dehn algorithm* for  $G$  over  $X$  if repeated application of these rules reduces any representative of the identity to the empty word. It is well-known that a group has a Dehn algorithm if and only if it is word-hyperbolic [1].

More generally (that is, even outside of the group theoretical context), if  $L$  is any set of strings over an alphabet  $X$  (or, in other words,  $L$  is any *language* over  $X$ ), we shall call  $L$   *$k$ -locally excluding* if there exists a finite set  $F$  of strings of length at most  $k$  such that a string  $w$  over  $X$  is in  $L$  if and only if  $w$  contains no substring in  $F$ . It is clear that the set of  $k$ -local geodesics in a group is  $k$ -locally excluding, since we can choose  $F$  to be the set of all non-geodesic words of length at most  $k$ . We observe in passing that if a set

of strings is  $k$ -locally excluding then, by definition, it is a  $k$ -locally testable and hence locally testable language (see [6]).

We shall say that the group  $G$  is  $k$ -locally excluding over a finite generating set  $X$  when the set of geodesics of  $G$  over  $X$  is  $k$ -locally excluding.

The purpose of this paper is to prove the following theorem.

**Theorem 1.** *Let  $G$  be a finitely generated group. Then the following are equivalent.*

- (i)  $G$  is virtually free.
- (ii) There exists a finite generating set  $X$  for  $G$  and a finite set of length-reducing rewrite rules over  $X$  whose application reduces any word over  $X$  to a geodesic word; that is  $G$  has a Dehn algorithm that reduces all words to geodesics.
- (iii) There exists a finite generating set  $X$  for  $G$  and an integer  $k$  such that every  $k$ -locally geodesic word over  $X$  is a geodesic; that is,  $G$  is  $k$ -locally excluding over  $X$ .

## 2. PROOF OF THEOREM 1

The equivalence of (ii) and (iii) is straightforward. Assume (ii), and let  $\mathcal{R}$  be a set of length-reducing rewrite rules with the specified property. Let  $k$  be the maximal length of a left hand side of a rule in  $\mathcal{R}$ . Then a  $k$ -local geodesic over  $X$  cannot have the left hand side of any rule in  $\mathcal{R}$  as a subword, and so it must be geodesic. Conversely, assume (iii) and let  $\mathcal{R}$  be the set of all rules  $u \rightarrow v$  in which  $l(v) < l(u) \leq k$  and  $u =_G v$ . Then repeated application of rules in  $\mathcal{R}$  reduces any word to a  $k$ -local geodesic which, by (iii), is a geodesic.

The main part of the proof consists in showing that (i) and (iii) are equivalent. We start with a useful lemma.

**Lemma 1.** *Let  $G$  be a group with finite generating set  $X$ , let  $k > 0$  be an integer, and suppose that  $G$  is  $k$ -locally excluding over  $X$ . Let  $w$  be a geodesic word over  $X$ , and let  $x \in X$ . Then*

- (i)  $l_G(wx)$  is equal to one of  $l(w) + 1$ ,  $l(w)$ ,  $l(w) - 1$ .
- (ii)  $wx$  is geodesic (that is,  $l_G(wx) = l(w) + 1$ ) if and only if  $vx$  is geodesic, where  $v$  is the suffix of  $w$  of length  $k - 1$  (or the whole of  $w$  if  $l(w) < k - 1$ ).
- (iii)  $l_G(wx) - l(w) = l_G(v'x) - l(v')$ , where  $v'$  is the suffix of  $w$  of length  $2k - 2$  (or the whole of  $w$  if  $l(w) < 2k - 2$ ).

*Proof.* The three possibilities for  $l_G(wx)$  follow from the fact that  $w$  is geodesic and  $x$  is a single generator. (ii) is an immediate consequence of  $G$  being  $k$ -locally excluding. (iii) follows from (ii) when  $wx$  is geodesic, so suppose not. Write  $w = uv$  with  $v$  as defined in (ii), and let  $z$  be a geodesic representative of  $vx$ . Since  $v$  is geodesic,  $l(z)$  is either  $l(v)$  or  $l(v) - 1$ . In the second case  $uz$  is geodesic, so  $l_G(wx) - l(w) = l_G(vx) - l(v) = l_G(v'x) - l(v') = -1$

and (iii) follows. In the first case ( $l(z) = l(v)$ ) write  $w = u'v''v$  with  $v' = v''v$ , so  $l(v'') = k - 1$  provided that  $u'$  is non-empty. Now  $wx = u'v''vx =_G u'v''z$  where  $l(u'v''z) = l(w)$ , and either  $l_G(wx) = l(u'v''z) = l(w)$  or  $l_G(wx) = l(u'v''z) - 1 = l(w) - 1$ . So at most one length reduction occurs in the word  $u'v''z$ , and since  $u'v''$  is geodesic, that length reduction must occur, if at all, within the subword  $v''z =_G v'x$ . Part (iii) follows from this.  $\square$

We are now ready to prove that (iii) implies (i) in Theorem 1.

**Proposition 1.** *Suppose that  $G$  is a group with finite generating set  $X$  and that the geodesics over  $X$  are  $k$ -locally excluding for some  $k > 0$ . Then  $G$  is virtually free.*

*Proof.* We prove this result by demonstrating that the word problem for  $G$  can be solved on a pushdown automaton, and then using Muller and Schupp's classification of groups with this property [5].

The automaton to solve the word problem operates as follows. Given an input word  $w$ , the automaton reads  $w$  from left to right. At any point, the word on the stack is a geodesic representative of the word read so far. Suppose at some point it has  $u$  on the stack and then reads a symbol  $x$ . It pops  $2k - 2$  symbols off the stack (or the whole of  $u$  if  $l(u) < 2k - 2$ ), appends  $x$  to the end of the word so obtained, replaces it by a geodesic representative if necessary, and appends that reduced word to the stack. It follows from Lemma 1 that the word now on the stack is a geodesic representative of  $ux$ , and hence of the word read so far.

So  $w$  represents the identity in  $G$  if and only if the stack is empty once all the input has been read and processed, and it follows immediately from [5] that  $G$  is virtually free.  $\square$

It remains to prove that (i) implies (iii), namely that the set of geodesics of a virtually free group with an appropriate generating set is  $k$ -locally excluding for some  $k > 0$ .

It is proved in [7, Theorem 7.3] that a finitely generated group  $G$  is virtually free if and only if it arises as follows:  $G$  is the fundamental group of a graph of groups  $\Gamma$  with finite vertex groups  $G_1, \dots, G_n$ , and finite edge groups  $G_{i,j}$  for certain pairs  $\{i, j\}$ .

There are various alternative and equivalent definitions of the fundamental group of a graph of groups, but the one that is most convenient for us is [2, Chapter 1, Definition 3.4]. As is pointed out in [2, Chapter 1, Example 3.5 (vi)], such a group  $G$  can be built up as a sequence of groups  $1 = H_1, H_2, \dots, H_r = G$ , where each  $H_{i+1}$  is defined either as a free product with amalgamation (over an edge group) of  $H_i$  with one of the vertex groups  $G_i$ , or as an HNN extension of  $H_i$  with associated subgroups isomorphic to one of the edge groups  $G_{i,j}$ . The amalgamated free products are done first, building up along a maximal tree, and then the HNN extensions are done for the remaining edges in the graph.

So from now on we shall assume that our virtually free group  $G$  can be constructed in this way, where the groups  $G_i$  and  $G_{i,j}$  are all finite. Hence the result follows from repeated application of the following two lemmas, of which the proofs are very similar.

Notice that the generating set  $X$  over which  $G$  is  $k$ -locally excluding will contain all non-identity elements of each of the vertex groups,  $G_i$  and also certain other elements arising from the HNN extensions, which are specified in Lemma 3.

**Lemma 2.** *Let  $H$  be a group which is  $k$ -locally excluding over a generating set  $X$  for some  $k \geq 2$ , let  $K$  be a finite group, let  $A = H \cap K$ , and suppose that  $A \setminus \{1\} \subset X$ .*

*Then  $G = H *_A K$  is  $k'$ -locally excluding over  $X' := X \cup (K \setminus A)$ , where  $k' = 3k - 2$ .*

**Lemma 3.** *Let  $H$  be a group which is  $k$ -locally excluding over a generating set  $X$  for some  $k \geq 2$ , let  $A$  and  $B$  be isomorphic finite subgroups of  $H$  which satisfy  $A \setminus \{1\} \subset X$  and  $B \setminus \{1\} \subset X$ , and let  $G = \langle H, t \rangle$  be the HNN extension in which  $tat^{-1} = \phi(a)$  for all  $a \in A$ , where  $\phi : A \rightarrow B$  is an isomorphism.*

*Then  $G$  is  $k'$ -locally excluding over  $X' := X \cup \{ta \mid a \in A\} \cup \{t^{-1}b \mid b \in B\}$ , where  $k' = 3k - 2$ . (Note that the elements of  $X'$  in the set  $\{t^{-1}b \mid b \in B\}$  are the inverses of those in the set  $\{ta \mid a \in A\}$ .)*

*Proof of Lemma 2.* Let  $w$  be a  $k'$ -local geodesic of  $G$  over  $X'$ . We want to prove that  $w$  is geodesic. Suppose not, and let  $w'$  be a geodesic word that represents the same element of  $G$ . Note that, since  $A \setminus \{1\} \subseteq X'$ , we cannot have  $w \in A$ , because that would imply that  $l(w) \leq 1$ .

We can write  $w = w_0 k_1 w_1 k_2 \cdots k_r w_r$ , where each  $k_i \in K \setminus A$  and each  $w_i \in X^*$ . Either  $w_0$  or  $w_r$  could be the empty word but, since  $K \setminus \{1\} \subseteq X'$  and  $w$  is a  $k'$ -local geodesic with  $k' > k \geq 2$ ,  $w_i$  must be non-empty for  $0 < i < r$ . The 2-locally excluding condition also implies that no non-empty  $w_i$  is a word in  $A^*$ . In fact, since  $H$  is by assumption  $k$ -locally excluding over  $X$  and  $k' > k$ , the words  $w_i$  are geodesics as elements of  $H$  over  $X$ , and so the non-empty  $w_i$  represent elements of  $H \setminus A$ .

Similarly, write  $w' = w'_0 k'_1 w'_1 k'_2 \cdots k'_{r'} w'_{r'}$ .

Now the normal form theorem for free products with amalgamation (see [4, Thm 4.4] or the remark following [3, Chapter 4, Theorem 2.6]) states that, if  $C$  is a union of sets of distinct right coset representatives of  $A$  in  $H$  and in  $K$ , then any element of the amalgamated product can be written uniquely as a product of the form  $ac_1 \cdots c_s$ , where  $a \in A$ , each  $c_i \in C$ , and alternate  $c_i$ 's are in  $H \setminus A$  and  $K \setminus A$ .

Since each  $k_i \in K \setminus A$  and each non-empty  $w_i \in H \setminus A$ , the syllable length  $s$  of the group element represented by  $w$  is equal to the number of non-trivial

words  $w_0, k_1, w_1, \dots, k_r, w_r$ , where  $c_1 \in H \setminus A$  if and only if  $w_0$  is non-trivial, and  $c_s \in H \setminus A$  if and only if  $w_r$  is non-trivial. The same applies to  $w'$ , and hence  $r = r'$ ,  $w_0$  and  $w'_0$  are either both empty or both non-empty, and similarly for  $w_r$  and  $w'_r$ .

Furthermore,  $w_r$  and  $w'_r$  are in the same right coset of  $A$  in  $H$ , and so  $w'_r =_H a_r w_r$  for some  $a_r \in A$ . Then  $k_r$  and  $k'_r a_r$  are in the same right coset of  $A$  in  $K$ , and so  $k_r =_K b_{r-1} k'_r a_r$  for some  $b_{r-1} \in A$ . Carrying on in this manner, we can show that there exist  $a_i, b_i \in A$  ( $0 \leq i \leq r$ ) such that  $w'_i =_H a_i w_i b_i$  and  $k'_i =_K b_{i-1}^{-1} k_i a_i^{-1}$ , where  $a_0 = b_r = 1$ .

Since  $r = r'$  and  $l(w') < l(w)$ , we must have  $l(w'_i) < l(w_i)$  for some  $i$ . So one of the words  $a_i w_i$ ,  $w_i b_i$ ,  $a_i w_i b_i$  must reduce (in  $H$  over  $X$ ) to a word strictly shorter than  $w_i$ .

Suppose first that  $w_i b_i$  reduces to a word strictly shorter than  $w_i$ . Since  $b_r = 1$ , we have  $i < r$  and so  $k_{i+1}$  exists. Then, by Lemma 1,  $l_H(v'_i b_i) = l(v'_i) - 1$ , where  $v'_i$  is the suffix of  $w_i$  of length  $2k - 2$ , or the whole of  $w_i$  if  $l(w_i) < 2k - 2$ . Now, since  $v'_i k_{i+1} =_G (v'_i b_i)(b_i^{-1} k_{i+1})$  with  $b_i^{-1} k_{i+1} \in K$ , we see that the suffix  $v'_i k_{i+1}$  of  $w_i k_{i+1}$ , which has length at most  $2k - 1$ , is a non-geodesic word in  $G$  and, since  $2k - 1 < k'$ , this contradicts the assumption that  $w$  is a  $k'$ -local geodesic.

The case in which  $a_i w_i$  reduces to a word of length less than  $w_i$  is similar (here we use a ‘mirror image’ of Lemma 1), and we find that  $i > 0$  and a prefix of  $k_i w_i$  of length at most  $2k - 1$  is non-geodesic, again contradicting the assumption that  $w$  is a  $k'$ -local geodesic.

It remains to consider the case where the reduction (in  $H$  over  $X$ ) of  $a_i w_i b_i$  is strictly shorter than  $w_i$ , but each of the reductions of  $a_i w_i$  and  $w_i b_i$  have the same length as  $w_i$ . Since neither  $a_i$  nor  $b_i$  can be trivial, we have  $0 < i < r$ , and so  $k_i$  and  $k_{i+1}$  both exist. We claim that  $w_i$  has length at most  $3k - 4$ . For if not, we write  $w_i = u' u v'$ , where  $l(u') = l(v') = k - 1$  and  $l(u) \geq k - 1$ , and deduce from Lemma 1 and its mirror image that  $a_i w_i b_i =_H y u z$ , where  $y, z \in X^*$  and  $l(y) = l(z) = k - 1$ . Then since  $y u z$  reduces in  $H$  over  $X$  and  $H$  is  $k$ -locally excluding over  $X$ , some subword of length  $k$  must reduce. Such a subword must be a subword of either  $y u$  or  $u z$ , and so one of  $a_i w_i$  or  $w_i b_i$  does indeed reduce to a word shorter than  $w_i$ , contradicting our assumption. Hence  $l(w_i) \leq 3k - 4$  as claimed.

Now  $k_i w_i k_{i+1}$  has length  $2 + l(w_i) \leq 3k - 2$ , but  $k_i w_i k_{i+1} =_G (k_i a_i^{-1}) w'_i (b_i^{-1} k_{i+1})$  with  $k_i a_i^{-1}, b_i^{-1} k_{i+1} \in K$ , so  $k_i w_i k_{i+1}$  is not a geodesic in  $G$  over  $X'$ , and once again we contradict our assumption that  $w$  is a  $k'$ -local geodesic. This completes the proof of Lemma 2.  $\square$

*Proof of Lemma 3.* Let  $w$  be a  $k'$ -local geodesic of  $G$  over  $X'$ . We want to prove that  $w$  is geodesic. Suppose not, and let  $w'$  be a geodesic word that represents the same element of  $G$ .

Write  $w = w_0 t_1^{\epsilon_1} w_1 t_2^{\epsilon_2} w_2 \cdots t_r^{\epsilon_r} w_r$ , where each  $t_i$  is one of the generators of the form  $ta$  ( $a \in A$ ), each  $\epsilon_i$  is 1 or  $-1$ , and each  $w_i$  is a word over  $X$ . Since  $k' > k$ ,  $w$  is a  $k$ -local geodesic, so each word  $w_i$  is geodesic as an element of  $H$ . So if  $w_i$  represents a non-trivial element of  $A$  or of  $B$ , then  $w_i$  has length 1. Hence, if  $\epsilon_i = 1$  then we cannot have  $w_i \in A \setminus \{1\}$ , and if  $\epsilon_i = -1$  then we cannot have  $w_i \in B \setminus \{1\}$ , because in those cases  $t^{\epsilon_i} w_i$  would be a non-geodesic subword of  $w$  of length 2. Also, if  $w_i$  is empty with  $0 < i < r$ , then  $\epsilon_i = \epsilon_{i+1}$ .

Similarly, write  $w' = w'_0 (t'_1)^{\epsilon'_1} w'_1 (t'_2)^{\epsilon'_2} w'_2 \cdots (t'_{r'})^{\epsilon'_{r'}} w'_{r'}$ .

Now the normal form theorem for HNN extensions [3, Chapter 4, Theorem 2.1] states that if  $C$  is a union of sets  $H_A$  and  $H_B$  of distinct right coset representatives of  $A$  and of  $B$  in  $H$ , then any element of the HNN extension  $G$  can be written uniquely as a product of the form  $ht^{\epsilon_1} c_1 \cdots t^{\epsilon_s} c_s$ , where  $h \in H$ , each  $\epsilon_i$  is 1 or  $-1$ , each  $c_i \in C$ , and  $c_i \in H_A$  or  $c_i \in H_B$  when  $\epsilon_i = 1$  or  $-1$ , respectively. Also, if  $c_i = 1$  with  $1 \leq i < s$ , then  $\epsilon_i = \epsilon_{i+1}$ .

For the normal form of the element of  $G$  represented by both  $w$  and  $w'$ , it follows that  $r = r' = s$  and  $\epsilon_i = \epsilon'_i = \epsilon_i$  for each  $i$ . Furthermore, an inductive argument similar to the one in the proof of Lemma 2 shows that there are elements  $a_i, b_i \in A \cup B$  ( $0 \leq i \leq r$ ) such that  $w'_i =_H a_i w_i b_i$  and  $(t'_i)^{\epsilon_i} = b_{i-1}^{-1} (t_i)^{\epsilon_i} a_i^{-1}$ , where  $a_0 = b_r = 1$ . We have  $a_i \in A$  or  $B$  when  $\epsilon_i = 1$  or  $-1$ , respectively, and  $b_i \in B$  or  $A$  when  $\epsilon_{i+1} = 1$  or  $-1$ , respectively.

Since  $r = r'$  and  $l(w') < l(w)$ , we must have  $l(w'_i) < l(w_i)$  for some  $i$ . So one of the words  $a_i w_i$ ,  $w_i b_i$ ,  $a_i w_i b_i$  must reduce (in  $H$  over  $X$ ) to a word strictly shorter than  $w_i$ .

Suppose first that  $w_i b_i$  reduces to a word strictly shorter than  $w_i$ . Since  $b_r = 1$ , we have  $i < r$  and so  $t_{i+1}$  exists. Then, by Lemma 1,  $l_H(v'_i b_i) = l(v'_i) - 1$ , where  $v'_i$  is the suffix of  $w_i$  of length  $2k - 2$ , or the whole of  $w_i$  if  $l(w_i) < 2k - 2$ . Now, since  $v'_i t_{i+1}^{\epsilon_{i+1}} =_G (v'_i b_i) (b_i^{-1} t_{i+1}^{\epsilon_{i+1}})$  with  $b_i^{-1} t_{i+1}^{\epsilon_{i+1}} \in X'$ , we see that the suffix  $v'_i t_{i+1}^{\epsilon_{i+1}}$  of  $w_i t_{i+1}^{\epsilon_{i+1}}$ , which has length at most  $2k - 1$ , is a non-geodesic word in  $G$  and, since  $2k - 1 < k'$ , this contradicts the assumption that  $w$  is a  $k'$ -local geodesic.

The case in which  $a_i w_i$  reduces to a word of length less than  $w_i$  is similar (using the mirror image of Lemma 1), and we find that  $i > 0$  and a prefix of  $t_i^{\epsilon_i} w_i$  of length at most  $2k - 1$  is non-geodesic, again contradicting the assumption that  $w$  is a  $k'$ -local geodesic.

It remains to consider the case where the reduction (in  $H$  over  $X$ ) of  $a_i w_i b_i$  is strictly shorter than  $w_i$ , but each of the reductions of  $a_i w_i$  and  $w_i b_i$  have the same length as  $w_i$ . Since neither  $a_i$  nor  $b_i$  can be trivial, we have  $0 < i < r$ , and so  $t_i$  and  $t_{i+1}$  both exist. We claim that  $w_i$  has length at most  $3k - 4$ . For if not, we write  $w_i = u'uv'$ , where  $l(u') = l(v') = k - 1$  and  $l(u) \geq k - 1$ , and deduce from Lemma 1 and its mirror image that  $a_i w_i b_i =_G yuz$ , where  $y, z \in X^*$  and  $l(y) = l(z) = k - 1$ . Then since  $yuz$  reduces in  $H$  over  $X$  and  $H$  is  $k$ -locally excluding over  $X$ , some subword of length  $k$  must reduce.

Such a subword must be a subword of either  $yu$  or  $uz$ , and so one of  $a_i w_i$  or  $w_i b_i$  does indeed reduce to a word shorter than  $w_i$ , contradicting our assumption. Hence  $l(w_i) \leq 3k - 4$  as claimed.

Now  $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}}$  has length  $2 + l(w_i) \leq 3k - 2$ , but  $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}} =_G (t_i^{\epsilon_i} a_i^{-1}) w'_i (b_i^{-1} t_{i+1}^{\epsilon_{i+1}})$  with  $l_G(t_i^{\epsilon_i} a_i^{-1}) = l_G(b_i^{-1} t_{i+1}^{\epsilon_{i+1}}) = 1$ , so  $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}}$  is not a geodesic in  $G$  over  $X'$ , and once again we contradict our assumption that  $w$  is a  $k'$ -local geodesic. This completes the proof of Lemma 3.  $\square$

## REFERENCES

- [1] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro and H. Short, *Notes on word-hyperbolic groups*, Proc. Conf. Group Theory from a Geometrical Viewpoint, eds. E. Ghys, A. Haefliger and A. Verjovsky, held in I.C.T.P., Trieste, March 1990, World Scientific, Singapore, 1991.
- [2] W. Dicks and M.J. Dunwoody, *Groups Acting on Graphs* Cambridge studies in advanced mathematics 17, Cambridge University Press, 1989.
- [3] R.C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
- [4] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Dover Publications Inc., New York, 1976.
- [5] David Muller and Paul E. Schupp, Groups, the theory of ends, and context-free languages, J. Comput. System Sci. 26, 1983, 295–310.
- [6] J.E. Pin, *Varieties of Formal Languages*, Plenum Publishing Corp., New York, 1986.
- [7] Peter Scott and Terry Wall, Topological methods in group theory. in *Homological group theory (Proc. Sympos., Durham, 1977)*, ed. C.T.C. Wall, London Math. Soc. Lecture Note Ser., 36, Cambridge Univ. Press, Cambridge-New York, 1979, 137–203.

DEPARTMENT OF MATHEMATICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN NJ 07030, USA

*E-mail address:* `rgilman@stevens-tech.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN NE 68588-0130, USA

*E-mail address:* `smh@math.unl.edu`

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UK

*E-mail address:* `dfh@maths.warwick.ac.uk`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEWCASTLE, NEWCASTLE NE1 7RU, UK

*E-mail address:* `Sarah.Rees@ncl.ac.uk`